Probability & Statistics

Tutorial #1

Academic Year 2025/2026

Outline

1. Problem 1: Set Operations with Finite Sets
2. Problem 2: Interval Operations on $\mathbb{R}$
3. Problem 3: Venn Diagrams
4. Problem 4: Sample Space for Coin Tosses
5. Problem 5: Divisibility and Inclusion-Exclusion
6. Problem 6: Partitions and Cardinality
7. Problem 7: Countability
8. Problem 12: Functions – Domain, Codomain, Range
9. Problem 13: Basic Probability
10. Problem 14: Probability with Set Operations
11. Problem 15: Dice Rolling Probabilities
12. Problem 16: Infinite Sample Space with Geometric Series
13. Problem 17: System of Equations for Probabilities
14. Summary

Problem 1 – Setup

Given:

- Universal set: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $A = \{1, 2, 3\}$
- $B = \{x \in S : 2 \leq x \leq 7\} = \{2, 3, 4, 5, 6, 7\}$
- $C = \{7, 8, 9, 10\}$

Problem 1a

Find: $A \cup B$

Recall:

- $A = \{1, 2, 3\}$
- $B = \{2, 3, 4, 5, 6, 7\}$

Problem 1a – Solution

Find: $A \cup B$

$A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5, 6, 7\}$

Union contains all elements in either set:

$$A \cup B = \boxed{\{1, 2, 3, 4, 5, 6, 7\}}$$

Problem 1b

Find: $(A \cup C) - B$

Recall:

- $A = \{1, 2, 3\}$
- $B = \{2, 3, 4, 5, 6, 7\}$
- $C = \{7, 8, 9, 10\}$

Problem 1b – Solution

Find: $(A \cup C) - B$

Step 1: Find $A \cup C$

$$A \cup C = \{1, 2, 3\} \cup \{7, 8, 9, 10\} = \{1, 2, 3, 7, 8, 9, 10\}$$

Step 2: Subtract B (remove elements that are in B)

$$(A \cup C) - B = \{1, 2, 3, 7, 8, 9, 10\} - \{2, 3, 4, 5, 6, 7\}$$

$$= \boxed{\{1, 8, 9, 10\}}$$

Problem 1c

Find: $\bar{A} \cup (B - C)$

Recall:

- $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- $A = \{1, 2, 3\}$
- $B = \{2, 3, 4, 5, 6, 7\}$
- $C = \{7, 8, 9, 10\}$

Problem 1c – Solution

Find: $\bar{A} \cup (B - C)$

Step 1: Find $\bar{A}$ (complement of A)

$$\bar{A} = S - A = \{4, 5, 6, 7, 8, 9, 10\}$$

Step 2: Find $B - C$

$$B - C = \{2, 3, 4, 5, 6, 7\} - \{7, 8, 9, 10\} = \{2, 3, 4, 5, 6\}$$

Step 3: Find the union

$$\bar{A} \cup (B - C) = \{4, 5, 6, 7, 8, 9, 10\} \cup \{2, 3, 4, 5, 6\}$$

$$= \boxed{\{2, 3, 4, 5, 6, 7, 8, 9, 10\}}$$

Problem 1d

Question: Do A, B, C form a partition of S?

Recall:

- $A = \{1, 2, 3\}$
- $B = \{2, 3, 4, 5, 6, 7\}$
- $C = \{7, 8, 9, 10\}$

Requirements for a partition:

1. Sets must cover S: $A \cup B \cup C = S$
2. Sets must be pairwise disjoint

Problem 1d – Solution

Do A, B, C form a partition of S?

Check pairwise disjointness:

- $A \cap B = \{1, 2, 3\} \cap \{2, 3, 4, 5, 6, 7\} = \{2, 3\} \neq \emptyset$ ✗
- $B \cap C = \{2, 3, 4, 5, 6, 7\} \cap \{7, 8, 9, 10\} = \{7\} \neq \emptyset$ ✗

The sets overlap!

No, they do NOT form a partition

Problem 2 – Setup

Universal set: $\mathbb{R}$ (real numbers)

Interval notation reminder:

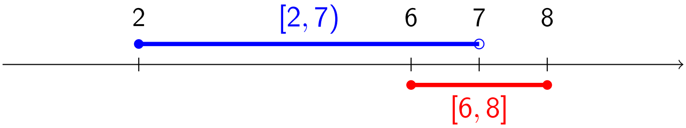
- $[a, b]$ = closed interval: $\{x \in \mathbb{R} : a \leq x \leq b\}$
- (a, b) = open interval: $\{x \in \mathbb{R} : a < x < b\}$
- $[a, b)$ = half-open: $\{x \in \mathbb{R} : a \leq x < b\}$

Problem 2a

Find: $[6, 8] \cup [2, 7)$

Problem 2a – Solution

Find: $[6, 8] \cup [2, 7)$



The intervals overlap from 6 to 7, and together cover from 2 to 8.

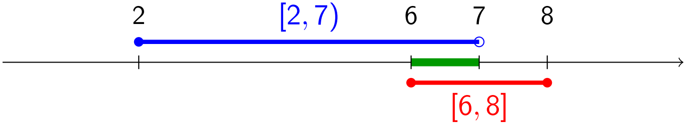
$$[6, 8] \cup [2, 7) = \boxed{[2, 8]}$$

Problem 2b

Find: $[6,8] \cap [2,7)$

Problem 2b – Solution

Find: $[6, 8] \cap [2, 7)$



The intersection is where both intervals overlap.
Note: 7 is NOT included (open in $[2, 7)$), but 6 IS included.

$$[6, 8] \cap [2, 7) = \boxed{[6, 7)}$$

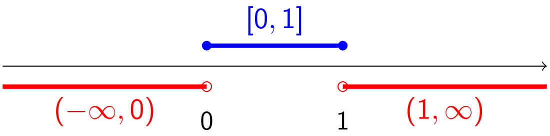
Problem 2c

Find: $[0, 1]^c$

Recall: The complement is with respect to $\mathbb{R}$

Problem 2c – Solution

Find: $[0,1]^c$



The complement is everything NOT in $[0,1]$:

$$[0,1]^c = \boxed{(-\infty, 0) \cup (1, \infty)}$$

Problem 2d

Find: $[6, 8] - (2, 7)$

Problem 2d – Solution

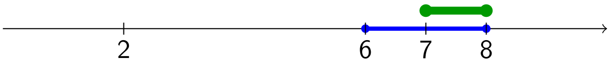
Find: $[6, 8] - (2, 7)$

Method: $[6, 8] - (2, 7) = [6, 8] \cap (2, 7)^c$

First, find $(2, 7)^c = (-\infty, 2] \cup [7, \infty)$

Then:

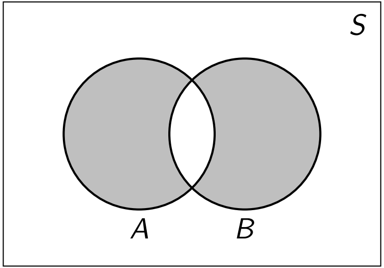
$$[6, 8] \cap ((-\infty, 2] \cup [7, \infty))$$



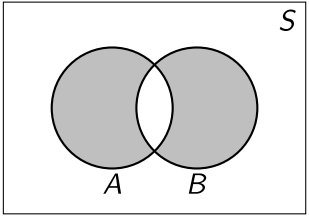
$$[6, 8] - (2, 7) = \boxed{[7, 8]}$$

Problem 3a

Write the set denoted by the shaded area:



Problem 3a – Solution



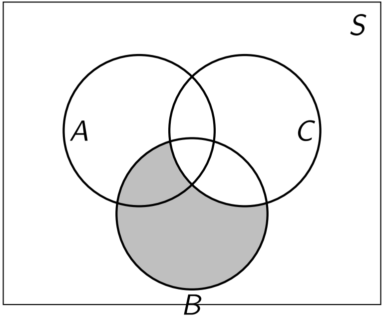
Step-by-step:

- The left shaded part is in A but not in B: this is $A - B$
- The right shaded part is in B but not in A: this is $B - A$
- The intersection $A \cap B$ is NOT shaded
- Combining both shaded parts: $(A - B) \cup (B - A)$

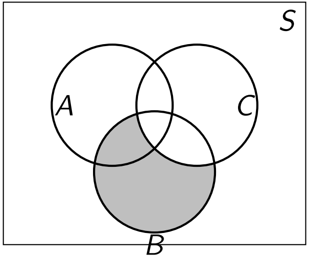
Answer: $\boxed{(A - B) \cup (B - A)}$

Problem 3b

Write the set denoted by the shaded area:



Problem 3b – Solution



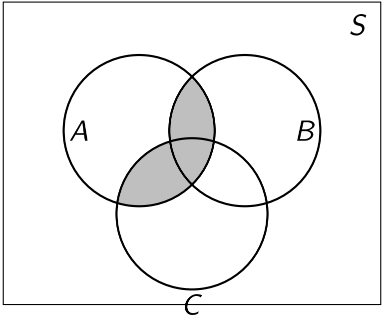
Step-by-step:

- The shaded region is entirely within circle B
- The part of B that overlaps with C is NOT shaded
- The part of B that overlaps with A IS shaded (it's not excluded)
- Therefore: elements in B but not in C

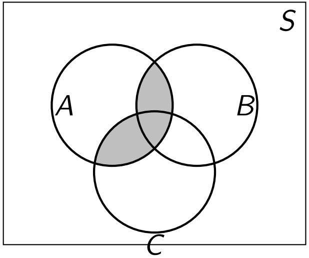
Answer: $\boxed{B - C}$

Problem 3c

Write the set denoted by the shaded area:



Problem 3c – Solution



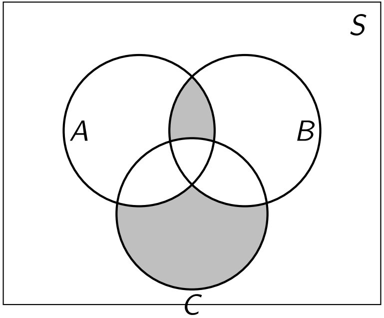
Step-by-step:

- One shaded part: inside both A and $B \Rightarrow A \cap B$
- Other shaded part: inside both A and $C \Rightarrow A \cap C$
- Combining both: $(A \cap B) \cup (A \cap C)$
- Note: this equals $A \cap (B \cup C)$ by distribution

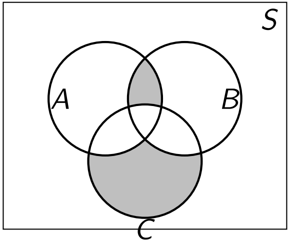
Answer: $\boxed{(A \cap B) \cup (A \cap C)}$

Problem 3d

Write the set denoted by the shaded area:



Problem 3d – Solution



Step-by-step:

- Bottom shaded region: in C, but not in A, and not in $B \Rightarrow C - A - B$
- Top shaded region: in both A and B, but not in $C \Rightarrow (A \cap B) - C$
- Combining both disjoint regions with union

Answer: $\boxed{(C - A - B) \cup ((A \cap B) - C)}$

Problem 4 – Setup

Setting: A coin is tossed twice.

Sample space:

$$S = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

Problem 4a

Find the set A: The first coin toss results in head.

Sample space: $S = \{(H, H), (H, T), (T, H), (T, T)\}$

List all outcomes where the first element is H.

Problem 4a – Solution

Find the set A: The first coin toss results in head.

Look for outcomes starting with H:

- (H, H) – first toss is H ✓
- (H, T) – first toss is H ✓
- (T, H) – first toss is T ✗
- (T, T) – first toss is T ✗

$$A = \boxed{\{(H, H), (H, T)\}}$$

Problem 4b

Find the set B: At least one tail is observed.

Sample space: $S = \{(H, H), (H, T), (T, H), (T, T)\}$

"At least one" means one or more.

Problem 4b – Solution

Find the set B**:** At least one tail is observed.

Check each outcome:

- (H, H) – zero tails ✗
- (H, T) – one tail ✓
- (T, H) – one tail ✓
- (T, T) – two tails ✓

$$B = \boxed{\{(H, T), (T, H), (T, T)\}}$$

Alternative: $B = S - \{(H, H)\}$ *(complement of "no tails")*

Problem 4c

Find the set C: The two coin tosses result in different outcomes.

Sample space: $S = \{(H, H), (H, T), (T, H), (T, T)\}$

Problem 4c – Solution

Find the set C: The two coin tosses result in different outcomes.

Check each outcome:

- (H, H) – same ✗
- (H, T) – different ✓
- (T, H) – different ✓
- (T, T) – same ✗

$$C = \boxed{\{(H, T), (T, H)\}}$$

Observation: $C = A \cap B$ in this problem!

Problem 5 – Setup

Given: $A = \{1, 2, \dots, 100\}$

For any $i \in \mathbb{N}$, define A_i as the set of numbers in A divisible by i.

Examples:

$$A_2 = \{2, 4, 6, \dots, 100\}$$
$$A_3 = \{3, 6, 9, \dots, 99\}$$

Key insight: $|A_i| = \lfloor 100/i \rfloor$ (floor function)

Problem 5a

Find: $|A_2|, |A_3|, |A_4|, |A_5|$

How many multiples of 2, 3, 4, and 5 are in $\{1, 2, \ldots, 100\}$*?*

Problem 5a – Solution

Find: $|A_2|, |A_3|, |A_4|, |A_5|$

Using $|A_i| = \lfloor 100/i \rfloor$:

$$|A_2| = \lfloor 100/2 \rfloor = \boxed{50}$$

$$|A_3| = \lfloor 100/3 \rfloor = \boxed{33}$$

$$|A_4| = \lfloor 100/4 \rfloor = \boxed{25}$$

$$|A_5| = \lfloor 100/5 \rfloor = \boxed{20}$$

Problem 5b

Find: $|A_2 \cup A_3 \cup A_5|$

How many numbers in $\{1, \ldots, 100\}$ *are divisible by 2, 3, or 5?*

Hint: Use the Inclusion-Exclusion Principle.

Problem 5b – Solution (Setup)

Inclusion-Exclusion Principle:

$$\begin{aligned}|A_2 \cup A_3 \cup A_5| = {} & |A_2| + |A_3| + |A_5| \\ & - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| \\ & + |A_2 \cap A_3 \cap A_5|\end{aligned}$$

Key observation: $A_i \cap A_j = A_{\text{lcm}(i,j)}$

Numbers divisible by both i and j are divisible by lcm(i, j)

Problem 5b – Solution (Calculation)

Calculate intersections:

- $|A_2 \cap A_3| = |A_6| = \lfloor 100/6 \rfloor = 16$
- $|A_2 \cap A_5| = |A_{10}| = \lfloor 100/10 \rfloor = 10$
- $|A_3 \cap A_5| = |A_{15}| = \lfloor 100/15 \rfloor = 6$
- $|A_2 \cap A_3 \cap A_5| = |A_{30}| = \lfloor 100/30 \rfloor = 3$

Apply Inclusion-Exclusion:

$$\begin{aligned} |A_2 \cup A_3 \cup A_5| &= 50 + 33 + 20 - 16 - 10 - 6 + 3 \\ &= \boxed{74} \end{aligned}$$

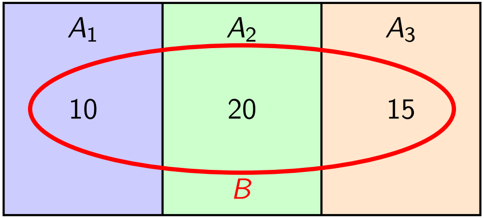
Problem 6

Given:

- A_1, A_2, A_3 form a partition of universal set S
- B is an arbitrary set
- $|B \cap A_1| = 10$
- $|B \cap A_2| = 20$
- $|B \cap A_3| = 15$

Find: $|B|$

Problem 6 – Solution



Key insight: Since A_1, A_2, A_3 partition S:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$$

These are **pairwise disjoint**, so:

$$|B| = |B \cap A_1| + |B \cap A_2| + |B \cap A_3| = 10 + 20 + 15 = \boxed{45}$$

Problem 7 – Setup

Determine whether each set is countable or uncountable.

Recall:

- *Countable* = finite OR can be put in bijection with $\mathbb{N}$
- *Uncountable* = cannot be enumerated (like $\mathbb{R}$)

Useful facts:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}$ is uncountable
- Subsets of countable sets are countable
- Cartesian products of countable sets are countable

Problem 7a

Is the following set countable or uncountable?

$$A = \{1, 2, \ldots, 10^{10}\}$$

Problem 7a – Solution

$$A = \{1, 2, \dots, 10^{10}\}$$

This is a **finite** set with 10^{10} elements.

(That's 10 billion elements – large but still finite!)

All finite sets are countable.

$$\boxed{\text{Countable}}$$

Problem 7b

Is the following set countable or uncountable?

$$B = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

Problem 7b – Solution

$$B = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

Each element of B is uniquely determined by a pair $(a, b) \in \mathbb{Q} \times \mathbb{Q}$.

Key facts:

- $\mathbb{Q}$ is countable
- $\mathbb{Q} \times \mathbb{Q}$ is countable (product of countable sets)

Since there's a surjection from $\mathbb{Q} \times \mathbb{Q}$ to B, the set B is at most countable.

$$\boxed{\text{Countable}}$$

Problem 7c

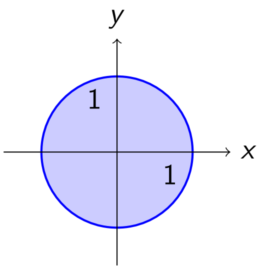
Is the following set countable or uncountable?

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

What does this set represent geometrically?

Problem 7c – Solution

$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
This is the **unit disk** (closed disk of radius 1 centered at origin).



This set contains the interval $[-1, 1]$ on the x-axis, which is uncountable.

$\boxed{\text{Uncountable}}$

Problem 12 – Setup

Given:

$$\{H,T\}^3 = \{H,T\} \times \{H,T\} \times \{H,T\}$$
$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Function: $f : \{H,T\}^3 \to \mathbb{N} \cup \{0\}$ defined as:

$$f(x) = \text{the number of H's in } x$$

Example: $f(HTH) = 2$

Determine the domain and codomain of f.

$$f : \{H, T\}^3 \to \mathbb{N} \cup \{0\}$$

Problem 12a – Solution

Determine the domain and codomain of f.

From the function definition $f : \{H, T\}^3 \to \mathbb{N} \cup \{0\}$:

Domain (set of inputs):

$$\boxed{\{H, T\}^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}}$$

Codomain (set of potential outputs):

$$\boxed{\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, \ldots\}}$$

Problem 12b

Find: Range(f)

The range is the set of actual output values.

What values can $f(x)$ actually take?

Problem 12b – Solution

Find: Range(f)

Calculate f for each input:

x	$f(x) = \#$ of H's
TTT	0
HTT, THT, TTH	1
HHT, HTH, THH	2
HHH	3

The actual output values are: Range(f) = $\boxed{\{0, 1, 2, 3\}}$

Note: Range $\subseteq$ Codomain, but Range $\neq$ Codomain here!

Problem 12c

If we know $f(x) = 2$, what can we say about x?

Find the preimage of 2 under f.

Problem 12c – Solution

If $f(x) = 2$, what can we say about x?

We need to find all x with exactly 2 heads:

- HHT – 2 heads ✓
- HTH – 2 heads ✓
- THH – 2 heads ✓

$$x \in \boxed{\{HHT, HTH, THH\}}$$

This is the **preimage** of 2: $f^{-1}(\{2\}) = \{HHT, HTH, THH\}$

Note: f is not injective (one-to-one) since multiple inputs give the same output.

Problem 13 – Setup

Setting: Two teams A and B play a soccer match.

Sample space:

$$S = \{a, b, d\}$$

where:

- a = Team A wins
- b = Team B wins
- d = Draw

Given:

- $P(a) = 0.5$
- $P(d) = 0.25$

Problem 13a

Find: The probability that B wins, i.e., $P(b)$.

Given:

- $P(a) = 0.5$
- $P(d) = 0.25$

Hint: What must probabilities sum to?

Problem 13a – Solution

Find: $P(b)$

Key principle: Probabilities must sum to 1 over the sample space.

$$P(a) + P(b) + P(d) = 1$$

Substitute and solve:

$$0.5 + P(b) + 0.25 = 1$$

$$P(b) = 1 - 0.5 - 0.25$$

$$P(b) = \boxed{0.25}$$

Problem 13b

Find: The probability that B wins or a draw occurs.

In set notation: $P(\{b, d\})$

Problem 13b – Solution

Find: $P(\{b, d\})$

Since $\{b\}$ and $\{d\}$ are **disjoint** events:

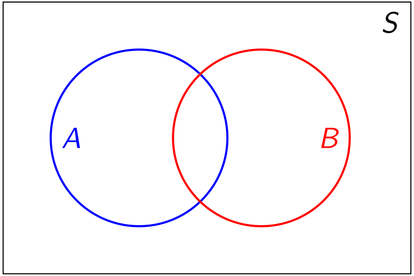
$$P(\{b, d\}) = P(b) + P(d) = 0.25 + 0.25 = \boxed{0.5}$$

Alternative approach using complement:

$$P(\{b, d\}) = P(\{a\}^c) = 1 - P(a) = 1 - 0.5 = 0.5$$

Problem 14 – Setup

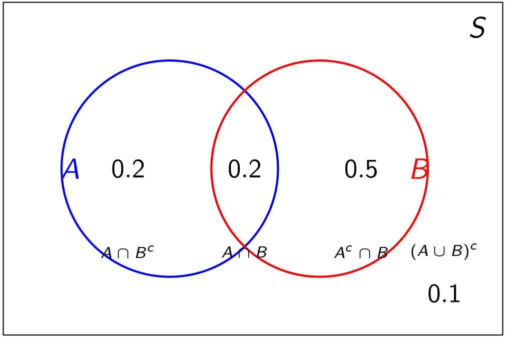
Given: Events A and B with: $P(A) = 0.4, \quad P(B) = 0.7, \quad P(A \cup B) = 0.9$



First, let's find $P(A \cap B)$ using Inclusion-Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.9 = 0.4 + 0.7 - P(A \cap B) \implies P(A \cap B) = 0.2$

Problem 14 – Probability Map



Summary of region probabilities:

- $P(A \cap B) = 0.2$
- $P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - 0.2 = 0.2$
- $P(A^c \cap B) = P(B) - P(A \cap B) = 0.7 - 0.2 = 0.5$
- $P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.9 = 0.1$

Problem 14a

Find: $P(A \cap B)$

Problem 14a – Solution

Find: $P(A \cap B)$

Using the Inclusion-Exclusion Principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute:

$$0.9 = 0.4 + 0.7 - P(A \cap B)$$
$$P(A \cap B) = 1.1 - 0.9 = \boxed{0.2}$$

Problem 14b

Find: $P(A^c \cap B)$

This is the part of B that is NOT in A.

Problem 14b – Solution

Find: $P(A^c \cap B)$

This is the part of B outside of A:

$$P(A^c \cap B) = P(B) - P(A \cap B)$$
$$= 0.7 - 0.2 = \boxed{0.5}$$

Problem 14c

Find: $P(A - B)$

Recall: $A - B = A \cap B^c$

Problem 14c – Solution

Find: $P(A - B) = P(A \cap B^c)$

This is the part of A outside of B:

$$P(A - B) = P(A) - P(A \cap B)$$
$$= 0.4 - 0.2 = \boxed{0.2}$$

Problem 14d

Find: $P(A^c - B)$

Hint: $A^c - B = A^c \cap B^c$

Problem 14d – Solution

Find: $P(A^c - B) = P(A^c \cap B^c)$

By De Morgan's Law: $A^c \cap B^c = (A \cup B)^c$

$$P(A^c - B) = P((A \cup B)^c) = 1 - P(A \cup B)$$
$$= 1 - 0.9 = \boxed{0.1}$$

Problem 14e

Find: $P(A^c \cup B)$

Problem 14e – Solution

Find: $P(A^c \cup B)$

Method 1: Using complement
Note that $(A^c \cup B)^c = A \cap B^c = A - B$

$$P(A^c \cup B) = 1 - P(A - B) = 1 - 0.2 = \boxed{0.8}$$

Method 2: Direct calculation

$$P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B)$$

$$= 0.6 + 0.7 - 0.5 = 0.8$$

Problem 14f

Find: $P(A \cap (B \cup A^c))$

Hint: Use the distributive law.

Problem 14f – Solution

Find: $P(A \cap (B \cup A^c))$

Using the distributive law:

$$A \cap (B \cup A^c) = (A \cap B) \cup (A \cap A^c)$$

Since $A \cap A^c = \emptyset$:

$$A \cap (B \cup A^c) = (A \cap B) \cup \emptyset = A \cap B$$

Therefore:

$$P(A \cap (B \cup A^c)) = P(A \cap B) = \boxed{0.2}$$

Problem 15 – Setup

Setting: Roll a fair die twice.

- X_1 = result of first roll
- X_2 = result of second roll

Sample space: $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

Total outcomes: $|S| = 6 \times 6 = 36$

Since die is fair: Each outcome has probability $\frac{1}{36}$

Problem 15a

Find: $P(X_2 = 4)$

What is the probability that the second roll is 4?

Problem 15a – Solution

Find: $P(X_2 = 4)$

Count favorable outcomes:

- X_1 can be anything: 6 choices
- $X_2 = 4$: 1 choice

Favorable outcomes: $(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)$ – that's 6 outcomes

$$P(X_2 = 4) = \frac{6}{36} = \boxed{\frac{1}{6}}$$

Note: This makes sense – the second roll is independent of the first!

Problem 15b

Find: $P(X_1 + X_2 = 7)$

What is the probability that the sum of both rolls is 7?

Problem 15b – Solution

Find: $P(X_1 + X_2 = 7)$

List all pairs (X_1, X_2) that sum to 7:

$$(1,6) \quad (2,5) \quad (3,4) \quad (4,3) \quad (5,2) \quad (6,1)$$

Favorable outcomes: 6

$$P(X_1 + X_2 = 7) = \frac{6}{36} = \boxed{\frac{1}{6}}$$

Problem 15c

Find: $P(X_1 \neq 2 \text{ and } X_2 \geq 4)$

Problem 15c – Solution

Find: $P(X_1 \neq 2 \text{ and } X_2 \geq 4)$

Count favorable outcomes:

- $X_1 \neq 2$: $X_1 \in \{1, 3, 4, 5, 6\}$ (5 choices)
- $X_2 \geq 4$: $X_2 \in \{4, 5, 6\}$ (3 choices)

Since the dice are independent:
Favorable outcomes: $5 \times 3 = 15$

$$P(X_1 \neq 2 \text{ and } X_2 \geq 4) = \frac{15}{36} = \boxed{\frac{5}{12}}$$

Alternative: $P(X_1 \neq 2) \cdot P(X_2 \geq 4) = \frac{5}{6} \cdot \frac{3}{6} = \frac{5}{12}$

Problem 16 – Setup

Given: Sample space $S = \{1, 2, 3, \ldots\}$ (positive integers)

Probability distribution: $P(k) = P(\{k\}) = \frac{c}{3^k}, \quad \text{for } k = 1, 2, 3, \ldots$
where c is a constant.

Problem 16a

Find: c

Hint: What must probabilities sum to over the entire sample space?

Problem 16a – Solution

Find: c

Probabilities must sum to 1: $\sum_{k=1}^{\infty} P(k) = 1 \implies \sum_{k=1}^{\infty} \frac{c}{3^k} = 1 \implies c \sum_{k=1}^{\infty} \frac{1}{3^k} = 1$

Geometric series: $\displaystyle\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$ for $|r| < 1$

With $r = \frac{1}{3}$: $\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$

Therefore: $c \cdot \frac{1}{2} = 1 \implies \boxed{c = 2}$

Problem 16b

Find: $P(\{2, 4, 6\})$

Now we know: $P(k) = \dfrac{2}{3^k}$

Problem 16b – Solution

Find: $P(\{2,4,6\})$

Since these are disjoint outcomes: $P(\{2,4,6\}) = P(2) + P(4) + P(6) = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6}$
$= \frac{2}{9} + \frac{2}{81} + \frac{2}{729}$

Finding common denominator (729): $= \frac{162}{729} + \frac{18}{729} + \frac{2}{729} = \frac{182}{729}$

$$P(\{2,4,6\}) = \boxed{\frac{182}{729}}$$

Problem 16c

Find: $P(\{3, 4, 5, \ldots\})$

Hint: Consider using the complement.

Problem 16c – Solution

Find: $P(\{3, 4, 5, \dots\})$

Using complement: $P(\{3, 4, 5, \dots\}) = 1 - P(\{1, 2\}) = 1 - P(1) - P(2) = 1 - \frac{2}{3} - \frac{2}{9}$
$= 1 - \frac{6}{9} - \frac{2}{9} = 1 - \frac{8}{9} = \boxed{\frac{1}{9}}$

Problem 17 – Setup

Setting: Four teams A, B, C, D compete. Exactly one wins.

Conditions:

1. Teams A and B have the same chance of winning
2. Team C is twice as likely to win as team D
3. $P(\text{A wins or C wins}) = 0.6$

Find: The probability of each team winning.

Problem 17 – Setting Up Equations

Let $P(A) = a$, $P(B) = b$, $P(C) = c$, $P(D) = d$

Translate conditions to equations:

1. Probabilities sum to 1: $a + b + c + d = 1$
2. A and B equal chance: $a = b$
3. C twice as likely as D: $c = 2d$
4. A or C wins with prob 0.6: $a + c = 0.6$

Problem 17 – Solve for Each Probability

Four equations:

$$a + b + c + d = 1 \quad (1)$$
$$a = b \quad (2)$$
$$c = 2d \quad (3)$$
$$a + c = 0.6 \quad (4)$$

Step 1: From (2): $b = a$
Step 2: From (3): $d = \frac{c}{2}$
Step 3: Substitute into (1): $a + a + c + \frac{c}{2} = 1 \implies 2a + \frac{3c}{2} = 1$

Problem 17 – Final Calculation

We have:

- $2a + \frac{3c}{2} = 1$
- $a + c = 0.6$

From the second equation: $a = 0.6 - c$

Substitute into the first: $2(0.6 - c) + \frac{3c}{2} = 1$ $1.2 - 2c + \frac{3c}{2} = 1$ $1.2 - \frac{c}{2} = 1$ $c = 0.4$

Problem 17 – Solution

Back-substitute to find all probabilities:

- $c = 0.4 \implies \boxed{P(C) = 0.4}$
- $a = 0.6 - c = 0.6 - 0.4 = 0.2 \implies \boxed{P(A) = 0.2}$
- $b = a = 0.2 \implies \boxed{P(B) = 0.2}$
- $d = \frac{c}{2} = \frac{0.4}{2} = 0.2 \implies \boxed{P(D) = 0.2}$

Problem 17 – Verification

Check all conditions:

$P(A) = 0.2,\ P(B) = 0.2,\ P(C) = 0.4,\ P(D) = 0.2$

1. Sum to 1? $0.2 + 0.2 + 0.4 + 0.2 = 1$ ✓
2. $P(A) = P(B)$? $0.2 = 0.2$ ✓
3. $P(C) = 2 \cdot P(D)$? $0.4 = 2 \times 0.2$ ✓
4. $P(A) + P(C) = 0.6$? $0.2 + 0.4 = 0.6$ ✓

All conditions satisfied!

Key Concepts Summary – Part 1 (Sets & Functions)

1. **Set Operations:** $\cup$ (union), $\cap$ (intersection), $-$ (difference), $\bar{A}$ (complement)
2. **Partition:** Sets that are pairwise disjoint and cover the universal set
3. **Interval Notation:** $[a, b]$ closed, (a, b) open, $[a, b)$ half-open
4. **Inclusion-Exclusion:**
   $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
5. **Countability:** Finite sets and sets bijective to $\mathbb{N}$ are countable
6. **Functions:** Domain (inputs), Codomain (potential outputs), Range (actual outputs)

Key Concepts Summary – Part 2 (Probability)

1. **Axioms of Probability:**
   - $P(A) \geq 0$ for all events
   - $P(S) = 1$
   - For disjoint events: $P(A \cup B) = P(A) + P(B)$
2. **Inclusion-Exclusion for Probability:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. **Complement Rule:** $P(A^c) = 1 - P(A)$
4. **Geometric Series:** $\displaystyle\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$ for $|r| < 1$
5. **Counting for Equally Likely Outcomes:** $P(A) = \frac{|A|}{|S|}$

Questions?

Slides will be published on
jkuwalek.github.io